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APPROXIMATION TECHNIQUES FOR CONTROL SYSTEMS WITH DELAYS

by

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Abstract

We present a theoretical framework for approximation techniques for nonlinear system optimal control problems. Two particular approximation schemes that may be used in the context of this framework are discussed and typical numerical results for two examples to which we have applied these schemes are given. We conclude with a brief survey of related investigations.

APPROXIMATION TECHNIQUES FOR CONTROL SYSTEMS WITH DELAYS

H. T. Banks and J. A. Burns

1. Introduction

In this report a modification of earlier approximation schemes (see [2,3,4,7,8]) for linear semigroups is used to develop a general framework for approximating certain nonlinear optimal control problems governed by functional differential equations (FDE). Particular types of schemes are discussed in the context of this framework, and numerical examples are given that illustrate some of our findings in using these schemes on "typical" control problems. We also present a short survey of recent and ongoing efforts which entail the application of semigroup approximation ideas to optimal control problems governed by FDE.

Throughout the paper, we shall denote by $L_2^{\nu}(a,b)$ (or $L_2(a,b)$ if $\nu = 1$) the customary Lebesgue space of R^{ν}-valued L_2 "functions" on (a,b). The symbol $|\cdot|$ will be used for all norms and it is to be understood that, in general, $|x|$ denotes the X norm of x whenever x is an element of the space X. The vector space of $n \times m$ matrices will be written as $\mathscr{L}_{n,m}$, and $Z = R^n \times L_2^n(-r,0)$ will be the usual product space. If $x\colon [-r,+\infty) \to R^n$ and $t \geq 0$, then the symbol x_t denotes the function $x_t\colon [-r,0] \to R^n$ given by $x_t(s) = x(t+s)$. Finally, we shall not, in general, distinguish between row and column vectors when the usage makes clear our intended meaning.

2. Representation of FDE by an abstract equation in Z

Let $D(t) \in \mathscr{L}_{n,m}$ be such that the function $t \to |D(t)|$ is in $L_2(-r,0)$, $0 = h_0 < h_1 < \ldots < h_\nu = r$, and $A_i \in \mathscr{L}_{n,n}$, $i = 1,2,\ldots,\nu$. If $\phi\colon [-r,0] \to R^n$, then define L to be the operator

$$L(\phi) = \sum_{i=0}^{\nu} A_i \phi(-h_i) + \int_{-r}^{0} D(\theta)\phi(\theta)d\theta.$$

We shall be concerned with the nonlinear control system

$$\begin{aligned} \dot{x}(t) &= L(x_t) + f(t,x(t),x_t,u(t)), \quad t \in [0,t_1] \\ x(0) &= \eta, \quad x_0 = \phi, \end{aligned} \tag{2.1}$$

where t_1 is finite, $(\eta,\phi) \in Z$, and f has the form

$$f(t,y,\psi,v) = \mathscr{N}_1(t,y,\psi) + \{ \mathscr{N}_2(t,y,\psi) + B(t)\}v.$$

We assume that the mapping $t \to B(t) \in \mathscr{L}_{n,m}$ is continuous, and $\mathscr{N}_1\colon R^1 \times Z \to R^n$, $\mathscr{N}_2\colon R^1 \times Z \to \mathscr{L}_{n,m}$ satisfy:

(A1) The mappings $(t,y,\psi) \to \mathscr{N}_i(t,y,\psi)$, $i = 1,2$, are continuous.

(A2) For each $\mathscr{D} \subseteq Z$ bounded, there exist bounded measurable functions K_1, K_2 such that

$$|\mathscr{N}_i(t,y,\psi) - \mathscr{N}_i(t,x,\phi)| \le K_i(t)|(y,\psi) - (x,\phi)|$$

for all (x,ϕ), $(y,\psi) \in \mathscr{D}$, $t \in R^1$, $i = 1,2$.

(A3) $\mathcal{N}_i(t,0,0) = 0$, $i = 1,2$, and there exist bounded measurable functions $\hat{K}_1, \hat{K}_2$ such that for $t \in R$,

$$|\mathcal{N}_i(t,y,\psi)| \leq \hat{K}_i(t)|(y,\psi)|, \quad i = 1,2,$$

for all (y,ψ) with $|(y,\psi)|$ sufficiently large.

For $t \geq 0$, define the operator $S(t): Z \to Z$ by $S(t)(\eta,\phi) = (x(t),x_t)$ where x is the solution to the homogeneous linear equation $\dot{x}(t) = L(x_t)$, satisfying $x(0) = \eta$, $x_0 = \phi$. It is known that $\{S(t)\}_{t\geq 0}$ is a linear C_0 semigroup on Z, and if $\mathcal{A}$ denotes the infinitesimal generator of this semigroup, then $\mathcal{A}$ is defined on $\mathcal{D}(\mathcal{A}) = \{(\eta,\phi) \mid \phi$ is absolutely continuous, $\dot{\phi} \in L_2^n(-r,0)$ and $\eta = \phi(0)\}$ by $\mathcal{A}(\eta,\phi) = (L(\phi),\dot{\phi})$. Let $\pi_1: Z \to R^n$ and $\pi_2: Z \to L_2^n(-r,0)$ be the coordinate projections, $\pi_1(\eta,\phi) = \eta$ and $\pi_2(\eta,\phi) = \phi$. Define $F: R^1 \times Z \times R^m \to Z$ by

$$F(t,z,v) = (f(t,\pi_1 z,\pi_2 z,v),0).$$

Consider the abstract integral equation in Z

$$z(t) = S(t)z_0 + \int_0^t S(t-\sigma)F(\sigma,z(\sigma),u(\sigma))d\sigma, \tag{2.2}$$

where $0 \leq t \leq t_1$ and $z_0 = (\eta,\phi) \in Z$. If f satisfies (A1)-(A3) and $u \in L_2^m(0,t_1)$, then it follows from the more general results established in [2] that equation (2.2) has a unique solution on

$[0,t_1]$. The following theorem is an extension of the result for linear systems (see [3]) and provides a slight improvement on the linear and nonlinear cases previously treated (see [2] - Theorem 2.1).

Theorem I. Suppose that f satisfies (A1)-(A3) and $u \in L_2^m(0,t_1)$. If $z_0 = (\eta,\phi)$ is any element in Z, then

$$z(t;z_0,u) = (x(t;z_0,u),\ x_t(z_0,u)), \tag{2.3}$$

where $t \to x(t;z_0,u)$ is the solution to (2.1) and $t \to z(t;z_0,u)$ is the solution to (2.2).

Observe that Theorem I differs from Theorem 2.1 in [2] in that we do not require that $z_0 = (\eta,\phi)$ belong to $\mathcal{D}(\mathcal{A})$. In particular, the equivalence (2.3) establishes unequivocally an abstract formulation for the control system (2.1) as the integral equation (2.2).

The proof of Theorem I follows the proof of Theorem 2.1 of [2] once one has established the continuity of the mappings $(z_0,u) \to (x(t;z_0,u),\ x_t(z_0,u))$ and $(z_0,u) \to z(t;z_0,u)$ from $Z \times L_2^m(0,t_1)$ into Z. Verification of these continuity requirements entail only slight modifications of the arguments behind Lemma 2.2 and Lemma 2.3 of [2].

Remark 2.1: Theorem I is the foundation for the general approximation technique to be detailed below. The idea is to approximate the linear semigroup $\{S(t)\}$ which will lead, via the equivalence (2.3), to approximations of the original nonlinear FDE system (2.1). Before pursuing this matter we note that Theorem I is also valid for the somewhat more general nonlinear systems considered in [2].

3. Approximating control systems

We now turn to development of our abstract framework for approximating optimal control problems governed by the nonlinear FDE system (2.1). This framework extends the linear theory given in [3] and is a slight modification of the ideas developed in [2]. Let $z(t;z_0,u)$ denote the solution to the abstract integral equation

$$z(t) = S(t)z_0 + \int_0^t S(t-\sigma)F(\sigma,z(\sigma),u(\sigma))d\sigma, \tag{3.1}$$

on the interval $[0,t_1]$. Our goal is to construct approximations to $z(t;z_0,u)$ and we shall do this by first approximating Z and $\{S(t)\}$.

We say that the sequence of quadruples $\{Z^N,P^N,M^N,S^N(t)\}$, $N = 1,2,\ldots$, is an <u>approximating sequence</u> if the following hypotheses are satisfied:

(H1) Z^N is a finite dimensional subspace of Z for each N.

(H2) $P^N: Z \to Z^N$ are (continuous) projections onto Z^N such that $\lim\limits_{N\to\infty}|P^N(\eta,\phi) - (\eta,\phi)| = 0$ for all $(\eta,\phi) \in Z$.

(H3) For each N, $M^N: Z^N \to Z^N$ is a linear operator, and there is a sequence $\{\gamma_N\}$ of constants such that $\lim\limits_{N\to\infty} \gamma_N = 0$

and $|M^N(\xi,0) - (\xi,0)| \le \gamma_N|\xi|$ for all $\xi \in R^n$.

(H4) The semigroups $\{S^N(t)\}$ satisfy $|S^N(t)| \le Ke^{\gamma t}$ (where K and γ are constants independent of N) and if $(\eta,\phi) \in Z$, then $\lim\limits_{N\to\infty}|S^N(t)(\eta,\phi) - S(t)(\eta,\phi)| = 0$ and the convergence is uniform in t on compact subsets of R^1.

Remark 3.1: The definition of an approximating sequence given above is related to but not the same as one employed by Trotter in [15]. The operators M^N were first introduced in [8] and they play an important role in certain approximation schemes (see the piecewise linear scheme in the next section and the discussions in [7]).

If $\{Z^N,P^N,M^N,S^N(t)\}$ is an approximating sequence, then we define for each N the approximating integral equation

$$z^N(t) = S^N(t)P^N z_0 + \int_0^t S^N(t-\sigma)M^N P^N F(\sigma,z^N(\sigma),u(\sigma))d\sigma, \tag{3.2}$$

where $0 \le t \le t_1$ and $z_0 = (\eta,\phi) \in Z$. As in [2], it can be shown that (3.2) has a unique solution for each $z_0 \in Z$ and $u \in L_2^m(0,t_1)$, provided f satisfies (A1)-(A3). Moreover, since Z^N is finite dimensional it follows that the integral equation (3.2) is equivalent to the ordinary differential system

$$\begin{aligned} \dot{z}^N(t) &= A^N z^N(t) + M^N P^N F(t,z^N(t),u(t)) \\ z^N(0) &= P^N z_0, \end{aligned} \tag{3.3}$$

where A^N is the infinitesimal generator for $\{S^N(t)\}$, i.e.,

$S^N(t) = e^{A^N t}$.

A few observations are now in order. First, in some schemes where elements of the form $(\xi, 0)$ are in Z^N (this is the case for the schemes to be discussed here - but is not the case in the event one considers the spline approximations in [6]) we point out that $F(t,z,v) = (f(t,\pi_1 z,\pi_2 z,v),0)$ is in Z^N. Hence, in (3.2) and (3.3) it is not necessary to project $F(t,z,v)$ onto Z^N with P^N. Secondly, the idea of approximating FDE systems by finite dimensional ordinary differential equations is not new and for a rather complete summary of previous efforts involving such techniques one may see [4].

Returning to the problems under discussion here, we observe that the following two theorems may be proven using the basic ideas in [2] and [7].

Theorem II. Let $\{Z^N, P^N, M^N, S^N(t)\}$ be an approximating sequence and suppose $\mathscr{U} \subseteq L_2^m(0,t_1)$ is bounded. If $z(t;z_0,u)$ and $z^N(t;z_0,u)$ are defined by (3.1) and (3.2), respectively, then $z^N(t;z_0,u) \to z(t;z_0,u)$ as $N \to \infty$, uniformly in t on $[0,t_1]$ and $u \in \mathscr{U}$.

Theorem III. Let $\{Z^N, P^N, M^N, S^N(t)\}$ be an approximating sequence. If $\{u^N\}$ is a sequence in $L_2^m(0,t_1)$ and $\{u^N\}$ converges weakly to u, then $z^N(t;z_0,u^N) \to z(t;z_0,u)$ as $N \to \infty$, uniformly in t on $[0,t_1]$.

Remark 3.2: Theorem III is the only result in this paper that does not extend to the more general nonlinear systems treated in [2]. The critical requirement in establishing this convergence is the weak continuity of the mapping $u \to z(t;z_0,u)$ from $L_2^m(0,t_1)$ into Z. However, for systems with nonlinearities satisfying (A1)-(A3) weak continuity can be argued (see [2] for more comments on this point).

If $z(t;z_0,u)$ and $z^N(t;z_0,u)$ are given by (3.1) and (3.2), respectively, then we let $z^N(t;z_0,u) = (x^N(t),y^N(t))$ (where $x^N(t) \in R^n$, $y^N(t) \in L_2^n(-r,0)$) and observe that since $z(t;z_0,u) = (x(t),x_t)$ we have convergence of $x^N(t)$ to $x(t)$ in R^n and $y^N(t)$ to x_t in $L_2^n(-n,0)$ (uniformly in t on $[0,t_1]$ and in $u \in \mathscr{E}$). Consider the optimal control problem:

($\mathscr{P}$) Minimize the cost

$$J(u) = \frac{1}{2}\, x(t_1)Gx(t_1) + \frac{1}{2}\int_0^{t_1} \{x(s)Qx(s) + u(s)Ru(s)\}ds$$

subject to (2.1) and $u \in \mathscr{U}$, where $\mathscr{U}$ is a closed convex subset of $L_2^m(0,t_1)$.

If $\{Z^N,P^N,M^N,S^N(t)\}$ is an approximating sequence, then we formulate the corresponding approximating control problems:

$(\mathscr{AP})^N$ Minimize the cost

$$J^N(u) = \frac{1}{2}\, x^N(t_1)Gx^N(t_1) + \frac{1}{2}\int_0^{t_1} \{x^N(s)Qx^N(s) + u(s)Ru(s)\}ds$$

subject to (3.2) and $u \in \mathscr{U}$.

Suppose now that for each N, $\bar{u}^N$ is an optimal control for the approximating problem $(\mathscr{AP})^N$ (note that $\bar{u}^N$ need not be unique in the general case of problems with a nonlinear system). Employing standard arguments (see Theorem 4.1 of [3]) one can show that $\{\bar{u}^N\}$ is a bounded sequence in $L_2^m(0,t_1)$ and possesses a subsequence which converges weakly to a control that yields a solution to problem $(\mathscr{P})$. Indeed, one finds

__Theorem IV__. Suppose that $\bar{u}^N$ is an optimal control for problem $(\mathscr{AP})^N$, $N = 1,2,\ldots$. Then there is an optimal control u^* for problem $(\mathscr{P})$ and a subsequence $\{\bar{u}^{N_k}\}$ such that $\{\bar{u}^{N_k}\}$ converges to u^* and $x^{N_k}(t;z_0,\bar{u}^{N_k}) \to x(t;z_0,u^*)$ as $N_k \to \infty$. Moreover, if u^* is the unique solution for $(\mathscr{P})$ then the original sequence $\{\bar{u}^N\}$ converges to u^* and $x^N(t;z_0,\bar{u}^N) \to x(t;z_0,u^*)$, uniformly in t.

4. Two approximation schemes

In this section, we outline two particular schemes that are included in the general framework presented in Section 3 and discuss some numerical examples based on these schemes. In order to facilitate exposition in illustrating these ideas, we consider the simplest case where the linear part of equation (2.1) has the form $L(\phi) = A_0\phi(0) + A_1\phi(-r)$.

Let $\{t_j^N\}$, $j = 0,1,2,\ldots,N$ be the partition of $[-r,0]$ defined by $t_j^N = \frac{-jr}{N}$ and let $\hat{t}_j^N$ be the midpoint of $[t_j^N,t_{j-1}^N] = I_j^N$. We let χ_j^N denote the characteristic function for $[t_j^N,t_{j-1}^N)$, while

$\psi_j^N(t) = (t-\hat{t}_j^N)\chi_j^N(t)$. If $(\eta,\phi) \in Z$, then for $j = 1,2,\ldots,N$ we define

$$\phi_j^N = \frac{N}{r}\int_{I_j^N} \phi(s)ds,$$

$$\hat{\phi}_j^N = \frac{12N^3}{r^3}\int_{I_j^N} (s-\hat{t}_j^N)\phi(s)ds,$$

and we define ϕ_0^N to be η.

The approximating sequence for what we shall call the <u>averaging approximations</u> is defined by quadruples

$$AVE = \{Z^N, P^N, M^N, S^N(t)\} \quad \text{where:}$$

$$Z^N = \{(\eta,\phi) \mid \eta \in R^n, \quad \phi = \sum_{j=1}^{N} v_j^N \chi_j^N, \ v_j^N \in R^n\};$$

$$P^N(\eta,\phi) = (\eta, \sum_{j=1}^{N} \phi_j^N \chi_j^N);$$

$M^N = I_N$, the identity on Z^N;

$S^N(t) = e^{A^N t}$, where A^N is the operator

$$A^N(\eta,\phi) = (A_0\eta + A_1\phi_N^N, \sum_{j=1}^{N} \frac{N}{r}\{\phi_{j-1}^N - \phi_j^N\}\chi_j^N).$$

The operator A^N may be loosely described as "the approximation of the infinitesimal generator $\mathscr{A}$ for $\{S(t)\}$ on the subspace Z^N" by using forward differences to approximate $\dot{\phi}$ (see [3], [4]). The proof that AVE is an approximating sequence may be found in [4].

The approximating sequence for what we have termed piecewise linear approximations is given by the quadruples

$$\text{PWL} = \{\hat{Z}^N, \hat{P}^N, \hat{M}^N, \hat{S}^N(t)\} \quad \text{where:}$$

$$\hat{Z}^N = \{(\eta,\phi) \mid \eta \in R^n,\ \phi = \sum_{j=1} v_j^N \chi_j^N + w_j^N \psi_j^N,\ v_j^N, w_j^N \in R^n\};$$

$$\hat{P}^N(\eta,\phi) = (\eta, \sum_{j=1}^{N} \phi_j^N \chi_j^N + \hat{\phi}_j^N \psi_j^N);$$

$$\hat{M}^N(\eta, \sum_{j=1}^{N} v_j^N \chi_j^N + w_j^N \psi_j^N) = (\eta, g), \text{ where}$$

$$g = \{\frac{1}{2}\eta + v_1^N\}\chi_1^N + \{\frac{N}{r}\eta + w_1^N\}\psi_1^N + \sum_{j=2}^{N} v_j^N \chi_j^N + w_j^N \psi_j^N\};$$

$$\hat{S}^N(t) = e^{\hat{A}^N t}.$$

Here $\hat{A}^N$ is the operator defined (again this involves a forward difference approximation for $\dot{\phi}$) by $\hat{A}^N(\eta,\phi) = (\xi,\psi)$ where $\xi = A_0\eta + A_1(\phi_N^N - \frac{r}{2N}\hat{\phi}_N^N)$ and $\psi(t) = \sum_{j=1}^{N} \alpha_j^N \chi_j^N + \beta_j^N \psi_j^N$ is determined by

$$\alpha_1^N = \frac{N}{r}(\eta - \phi_1^N) + \frac{A_0}{2}\eta + \frac{A_1}{2}(\phi_N^N - \frac{r}{2N}\hat{\phi}_N^N),$$

$$\beta_1^N = \frac{N}{r}\{A_0\eta + A_1(\phi_N^N - \frac{r}{2N}\hat{\phi}_N^N) - \hat{\phi}_1^N\},$$

$$\alpha_j^N = \frac{N}{r}(\phi_{j-1}^N - \phi_j^N),\ \beta_j^N = \frac{N}{r}(\hat{\phi}_{j-1}^N - \hat{\phi}_j^N),\quad j = 2,\dots,N.$$

The proof that PWL is an approximating sequence is given in [7].

We present next a summary of two examples that are typical of the behavior we have observed in using the above schemes in numerical computations. For results of tests with numerous other examples (both linear and nonlinear) one should consult [2], [4], [5], [7], [8].

Example 1: We consider the problem of minimizing

$$J(u) = \frac{2}{3}\, x(2)^2 + \frac{1}{2}\int_0^2 [u(t)]^2 dt$$

over $u \in \mathscr{U} = L_2(0,2)$ subject to

$$\dot{x}(t) = x(t-1) + u(t), \qquad 0 \le t \le 2,$$
$$x(\theta) = \phi(\theta), \qquad -1 \le \theta \le 0,$$

where

$$\phi(\theta) = \begin{cases} 2\theta & -1 \le \theta < -\frac{1}{2}, \\ -2\theta + 1 & -\frac{1}{2} \le \theta \le 0 \end{cases} .$$

One can use necessary conditions for delay system control problems to find an analytical solution to this problem (for details, see [7]). The optimal control is given by

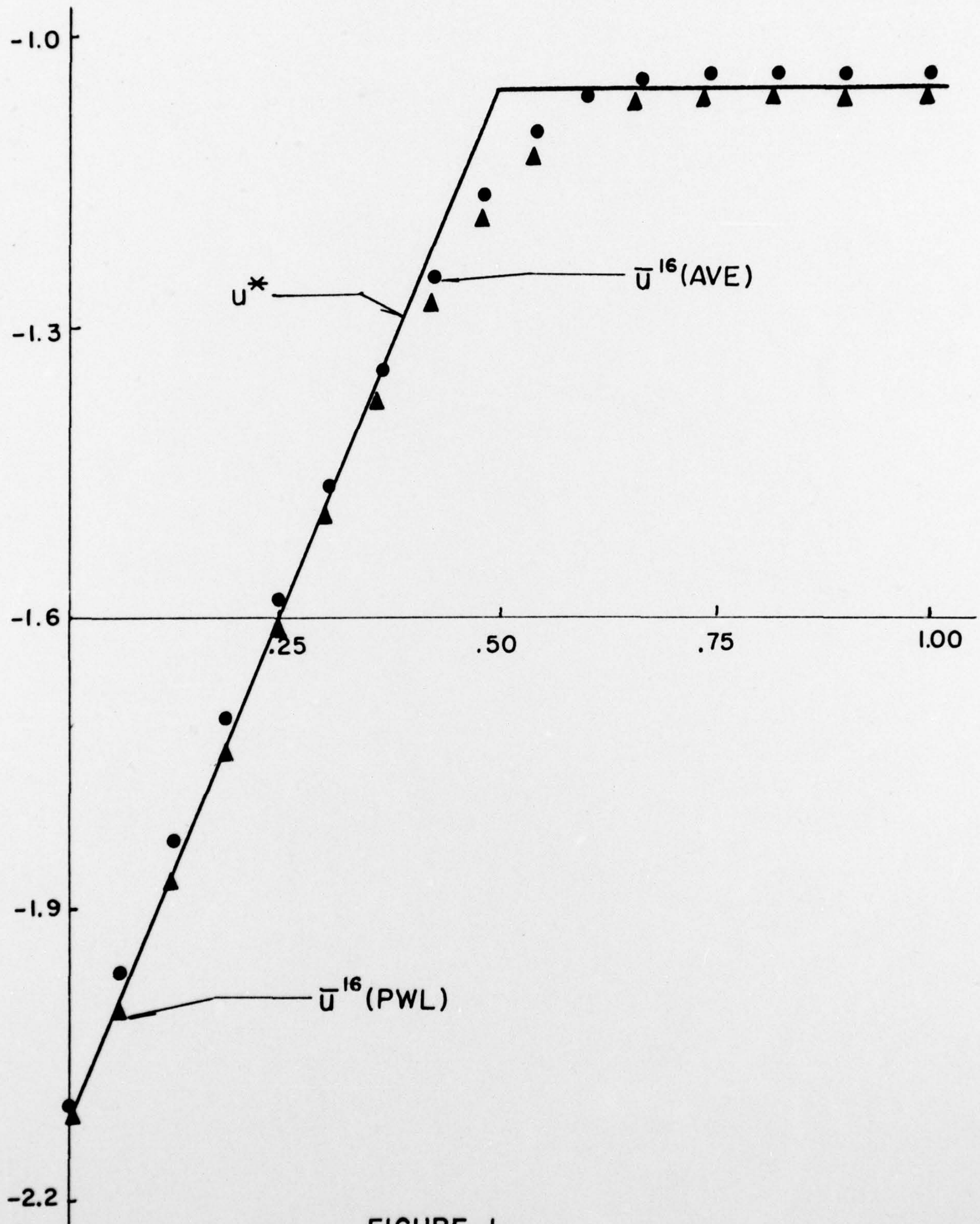

FIGURE 1

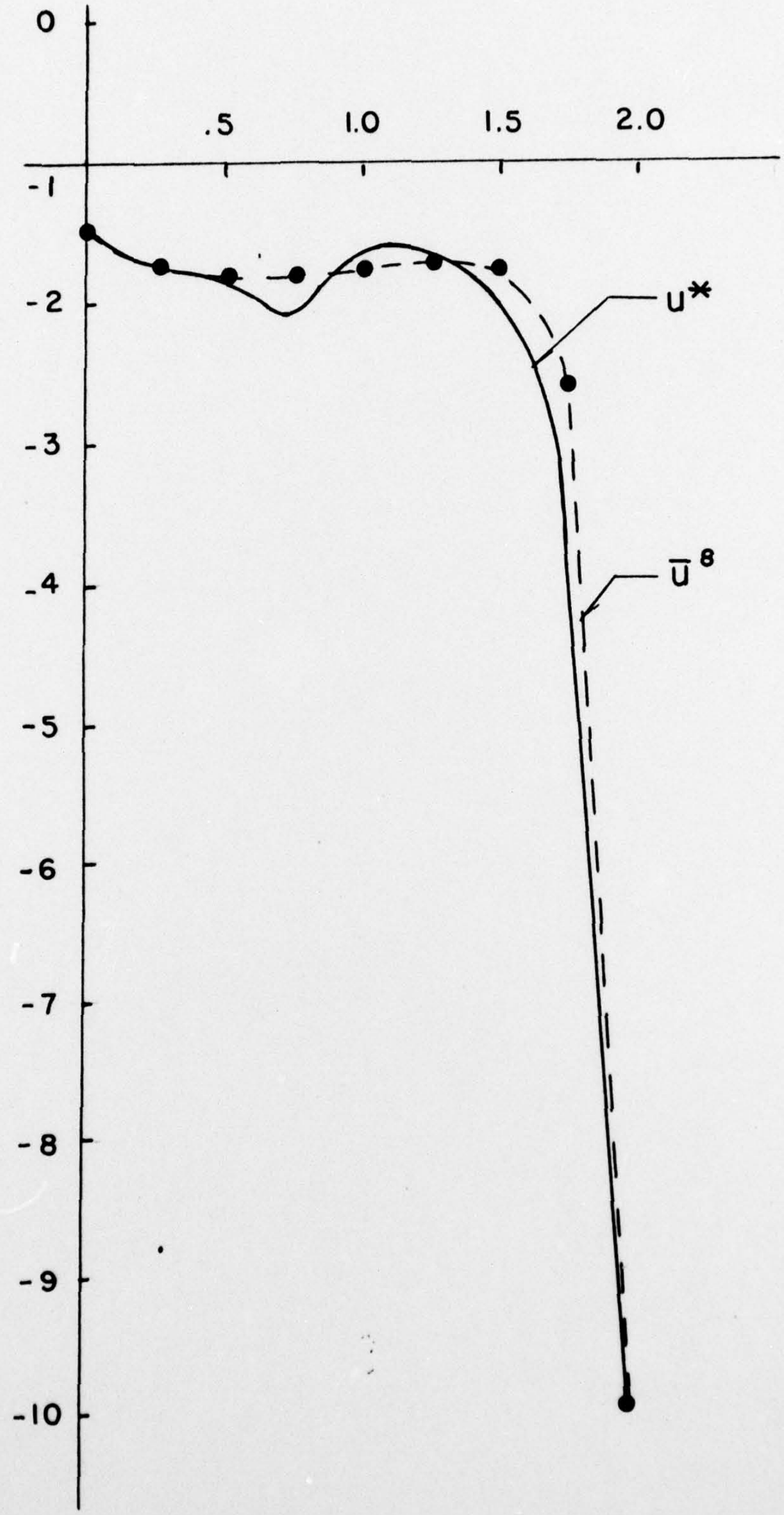

FIGURE 2

$$u^*(t) = \begin{cases} \delta t - 2\delta & 0 \le t \le 1, \\ -\delta & 1 \le t \le 2, \end{cases}$$

where $\delta \approx 1.05102$, and the corresponding value of the payoff is $J(u^*) \approx 2.55531$. In Figure 1, we plot typical approximate controls (for $N = 16$) for the AVE and PWL schemes described above, along with a plot of u^*.

Example 2: The problem is to minimize

$$J(u) = \frac{1}{2} x(2)^2 + \frac{1}{2} \int_0^2 \{x(t)^2 + u(t)^2\} dt$$

over $u \in \mathcal{U} = L_2(0,2)$ subject to the nonlinear system

$$\dot{x}(t) = x(t) \sin x(t) + x(t-1) + u(t), \qquad 0 \le t \le 2,$$
$$x(\theta) = 10, \qquad -1 \le \theta \le 0.$$

While we are not able to solve this problem analytically, we can use directly the necessary conditions for delay systems to get a numerical approximation for u^* (see [2]). This can, in turn, be used to check the convergence of schemes such as those under discussion here. Figure 2 contains plots of $\bar{u}^8$ (using the AVE scheme) and the independently obtained numerical approximation for u^*.

5. Final remarks.

We conclude with brief comments on recent efforts directly related to those reported here. First, complete and detailed discussions (along with error estimates, numerical results for a number of other examples, and remarks on the role of the operators M^N) for the averaging (AVE) and piecewise linear (PWL) schemes detailed above may be found in [4], [7], and [8]. Other recent contributions involving the use of a semigroup framework for approximation methods include the work of Reber in [11] and Banks and Kappel in [6]. Reber, employing factor space methods (see Chapter 5 of [10]) has carried out theoretical and numerical investigations for schemes that entail simultaneous discretizations (in both the time and state variables). This underlying idea, which leads directly to finite dimensional difference equation control problems (as opposed to the ordinary differential equation approximating problems such as those discussed above), was used in [11] to develop a theoretical framework for problems with general nonautonomous linear FDE control systems. In [6] the authors show that the framework developed in [2], [3], and [4] is an appropriate setting for the use of splines in approximating linear and nonlinear FDE.

Several other investigators have used semigroups to treat nonlinear FDE in the spirit of the framework given in [3]. Reddien and Webb [12] and Sasai and Ishigaki [13], assuming a global Lipschitz condition on the nonlinearities of the system, use ideas from nonlinear semigroup theory (as developed in recent years in the investigation of nonlinear evolution equations in Hilbert spaces) to obtain convergence results for approximation schemes of the averaging type discussed above. Kappel and Schappacher [9] develop a "local" nonlinear semigroup theory which requires only a local Lipschitz condition on their autonomous nonlinear FDE and obtain approximation techniques which are also of the averaging type. To date, only Sasai has considered any of these nonlinear semigroup results in the context of optimal control problems and the theoretical results given in [14] appear to be applicable in practice only to a much more restricted class of control problems (problems with admissible sets of smooth controls that are essentially conditionally compact) than those discussed above or in [2].

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